Mem. Fac. Sci.
Kochi Univ. (Math.)
11 (1990), 45–64

TYPE I DEGENERATION OF KUNEV SURFACES. II

Sampei Usui

(Received September 30, 1989)

Introduction

This is a continuation of [Us. 6].

§5 contains the tables of the global classification of the branch loci and the canonical curves of the main components of the central fibers of all the degenerations of Kunev surfaces with finite local monodromy on the second cohomology. The result here is clumsy but fruitful. Besides of another elementary proof of the main theorem [Us.6, (2.6.3)] given in (5.3), we can observe, for example, series of degenerations of the canonical curves in each case by Tables in (5.2). This is interesting, since we are concerned about degenerations of pairs.

We shall use the results and the references in [Us.6] freely.

Because of the reason of publication, we separate the present part, §5, from [Us.6].

§5. Global computation of branch locus B_Y on Y and the canonical curve $K_{\hat{X}}$ on $\hat{X}$.

(5.0) We use the notation in (3.0) and (3.1) throughout this section. In this section we shall compute globally the branch locus B_Y on the minimal K3 surface Y and the canonical curve $K_{\hat{X}}$ of the minimal model $\hat{X}$ of the fiber $X = X_t = f^{-1}(t)$, $t \in U$.

(5.1) We devide the distinguished (-2)-curves E_i $(1 \leq i \leq 9)$ on Y (cf. Table (3.2.2)) into two types:

Type I. $\alpha_1(E_i)$ is not on L.

Type II. $\alpha_1(E_i)$ is on L.

Then the branch locus B_Y on Y in (3.1.3) is divided into

(5.1.1) $$B_Y = B_Y(I) + B_Y(L)$$

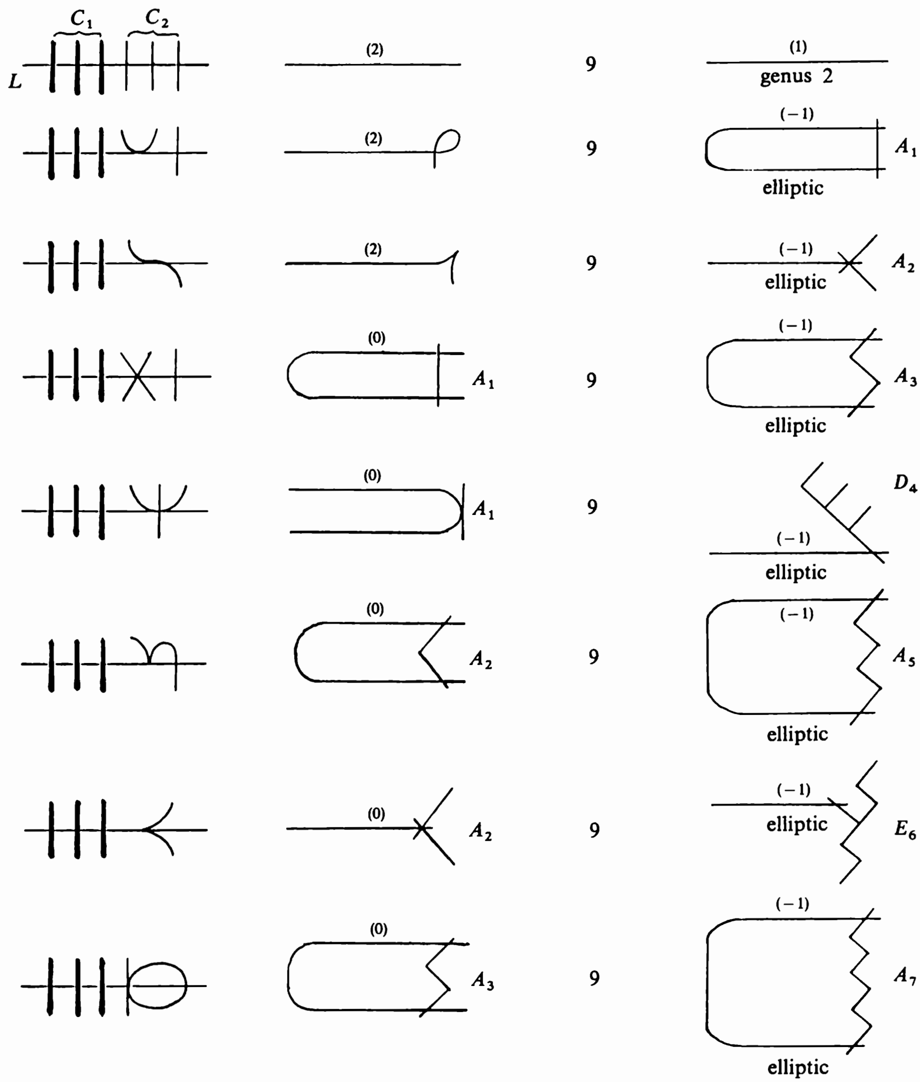
where $B_Y(I)$ is the reduced divisor consisting of the mutually disjoint (-2)-curves of Type I and $B_Y(L)$ is the reduced divisor consisting of the components of B_Y which are mapped to L by α_1. Notice that $B_Y(I)$ is disjoint from $B_Y(L)$ and become mutually disjoint (-1)-curves on the canonical resolution X^* and contract to points on $\hat{X}_1$.

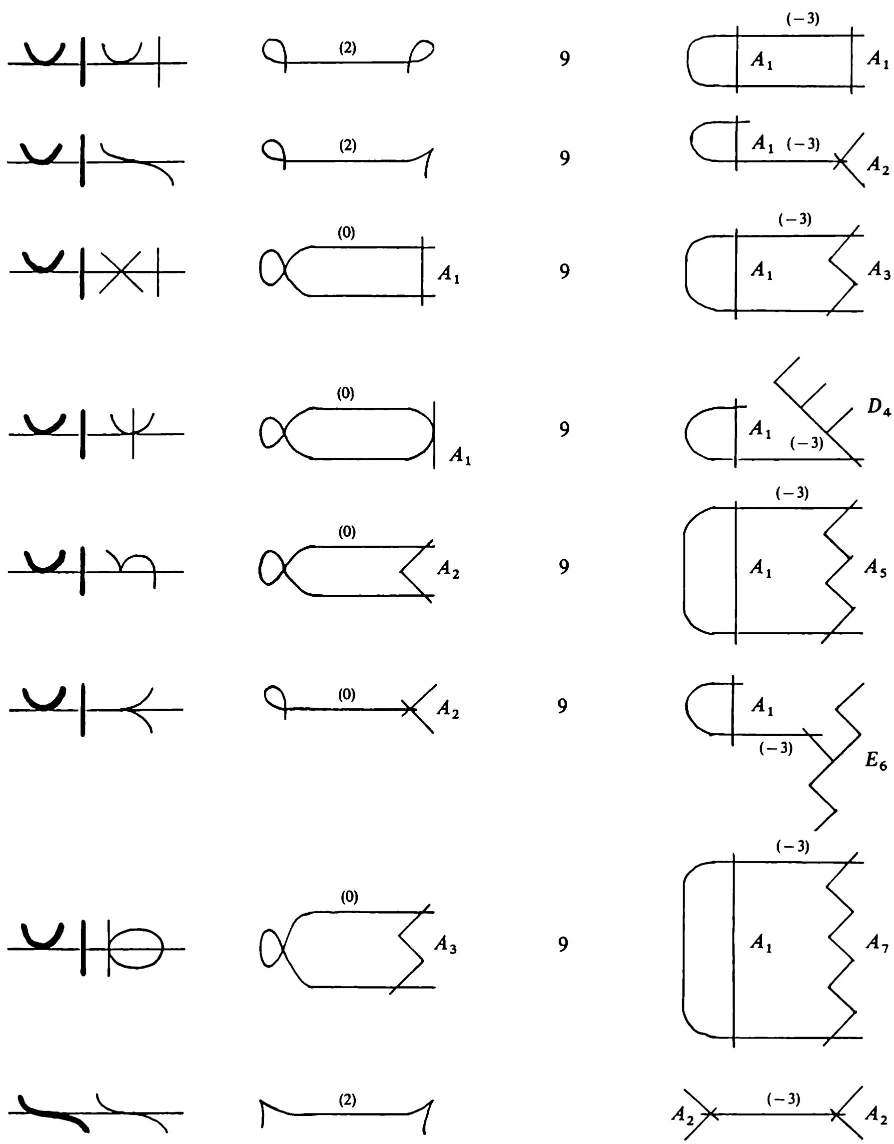
Partially supported by Grant-in-Aid for Composite Scientific Research B-01306001 and General Scientific Research C-01540060 from the Ministry Education of Japan.

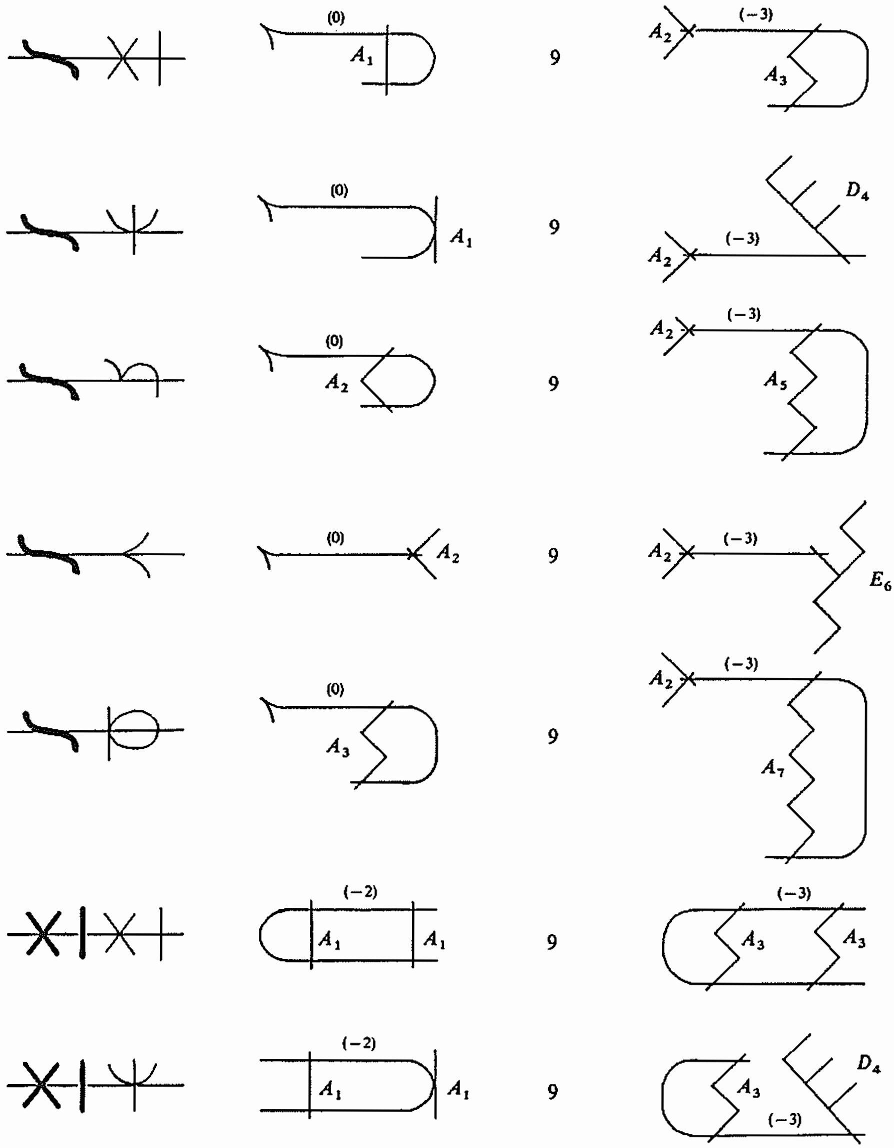
(5.2) As for $B_Y(L)$, we can compute it following the procedure of Diagram (3.1.2). Each process is elementary. We give here the tables of configurations of the two cubics and the line $\sum C_j + L$ on $\mathbf{P}^2$, the divisor $B_Y(L)$ and the number of the components $\# B_Y(I)$ of the divisor $B_Y(I)$ on the minimal K3 surface Y (see (5.1.1)), and the canonical divisor $K_{\hat{X}}$ of the minimal model $\hat{X}$ of X, which is always not multiple by Observation (3.3.1).

(5.2.1) Case $m(\sum C_j, L) = n(\sum C_j, L) = 0$ (36 types):

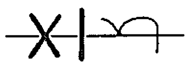
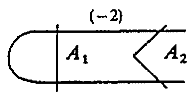
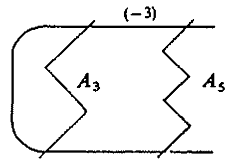
on P^2, C_1: bold curves	$B_Y(L)$, ( ): self-intersection	$\# B_Y(I)$	$K_{\hat{X}}$, ( ): self-intersection

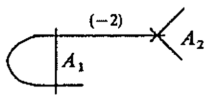
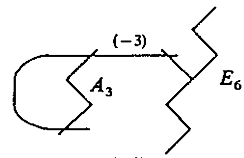


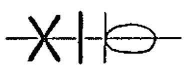
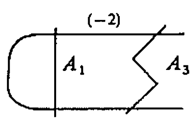
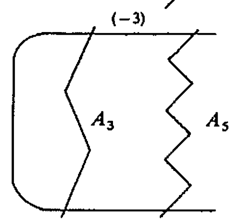

(2)
9
(−3)
A_1
A_1
(2)
9
A_1 (−3)
A_2
(0)
A_1
9
(−3)
A_1
A_3
(0)
A_1
9
D_4
A_1
(−3)
(0)
A_2
9
(−3)
A_1
A_5
(0)
A_2
9
A_1
(−3)
E_6
(0)
A_3
9
(−3)
A_1
A_7
(2)
A_2
(−3)
A_2

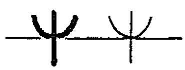
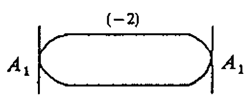
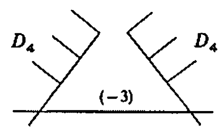


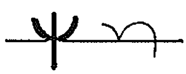
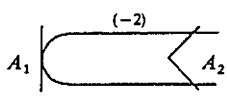
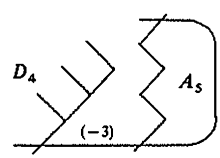
(0)
A_1
9
(−3)
A_2
A_3
(0)
A_1
9
D_4
(−3)
A_2
(0)
A_2
9
(−3)
A_2
A_5
(0)
A_2
9
(−3)
A_2
E_6
(0)
A_3
9
(−3)
A_2
A_7
(−2)
A_1
A_1
9
(−3)
A_3
A_3
(−2)
A_1
A_1
9
D_4
A_3
(−3)

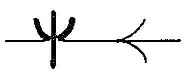
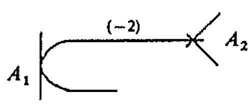
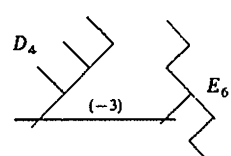


(−2)
A_1
A_2
9

(−3)
A_3
A_5

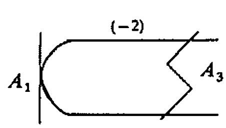
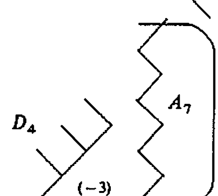


(−2)
A_2
A_1
9

(−3)
A_3
E_6

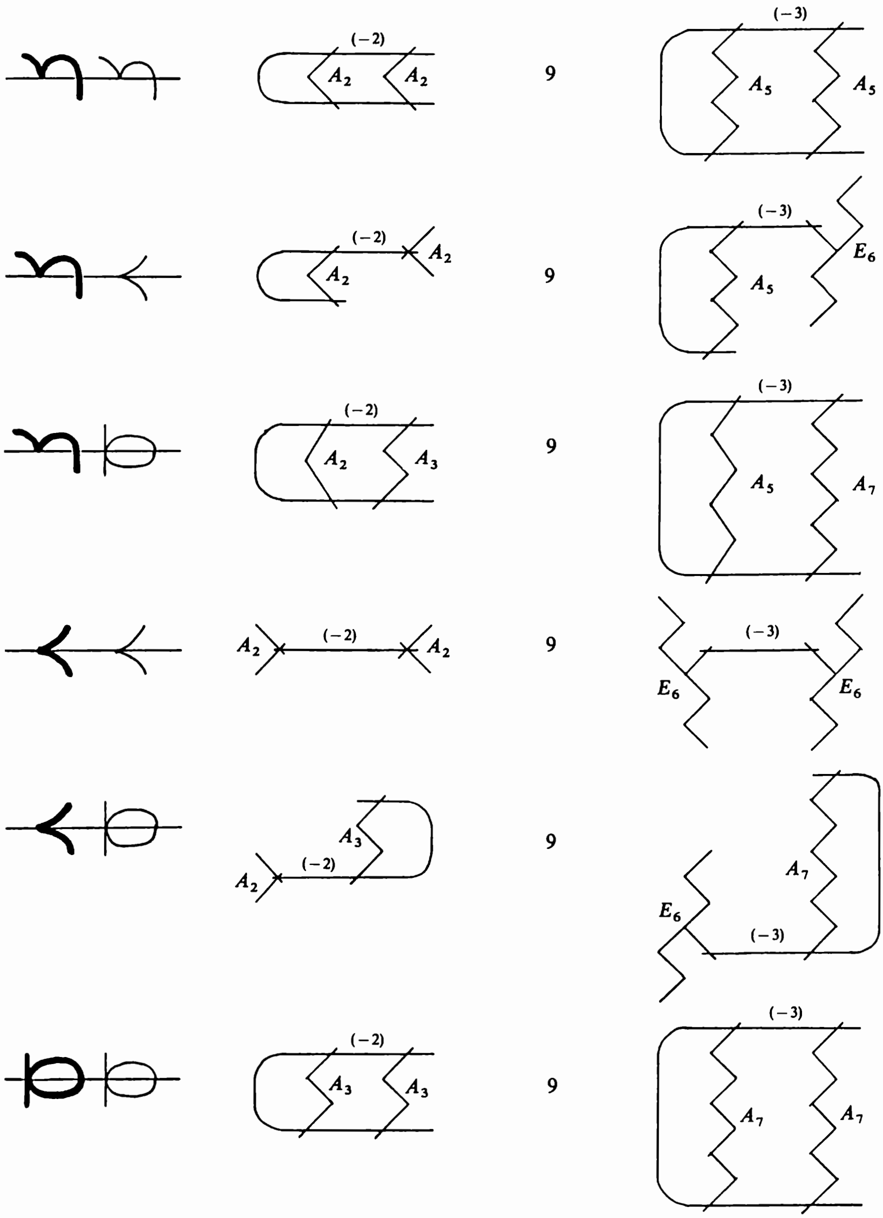


(−2)
A_1
A_3
9

(−3)
A_3
A_5



(−2)
A_1
A_1
9

D_4
D_4
(−3)



(−2)
A_1
A_2
9

D_4
A_5
(−3)



(−2)
A_2
A_1
9

D_4
E_6
(−3)



(−2)
A_1
A_3
9

A_7
D_4
(−3)



(−2)
A_2 A_2
9
(−3)
A_5 A_5
(−2)
A_2 A_2
9
(−3)
E_6
A_5
(−2)
A_2 A_3
9
(−3)
A_5 A_7
A_2 (−2) A_2
9
(−3)
E_6 E_6
A_3
(−2)
A_2
9
A_7
E_6
(−3)
(−2)
A_3 A_3
9
(−3)
A_7 A_7

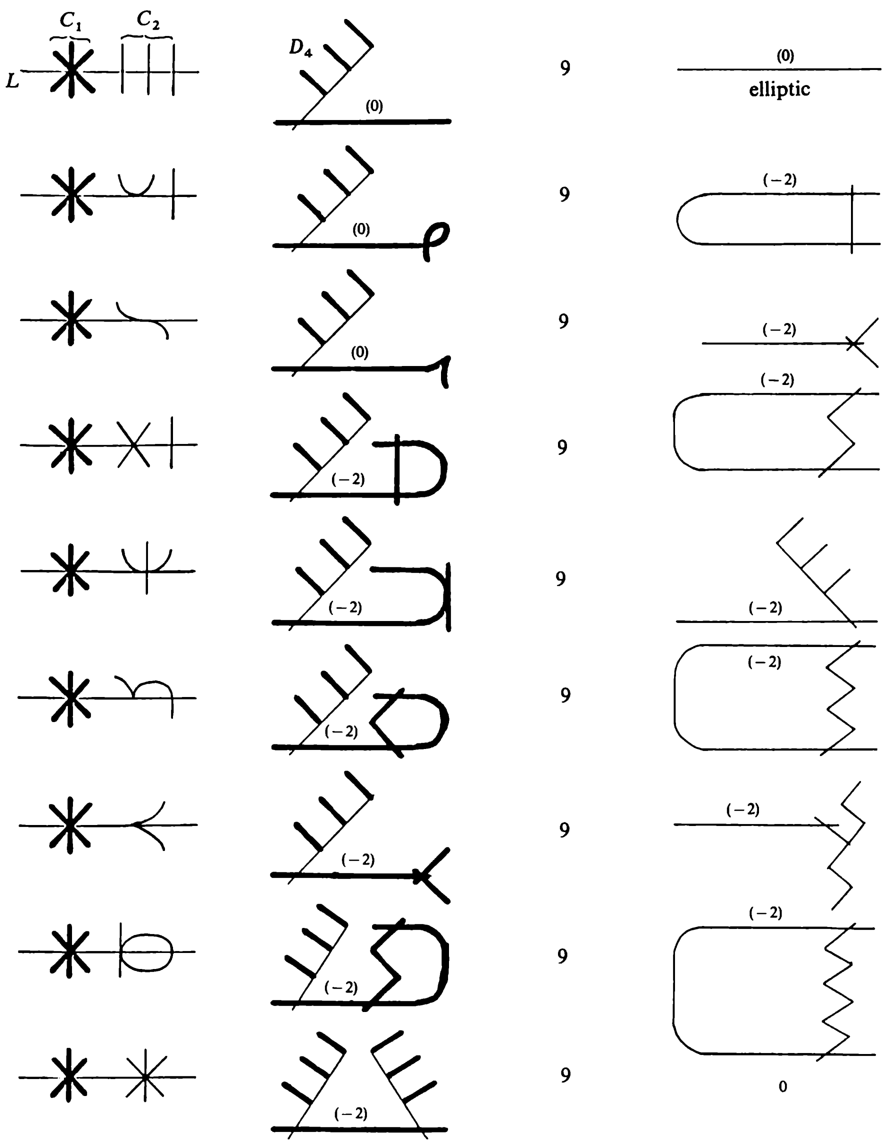
(5.2.2) Case $m(\sum C_j, L) = 0$, $n(\sum C_j, L) > 0$ (8 + 1 = 9 types):

on P^2,
C_1: bold curves

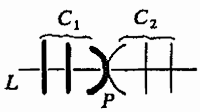
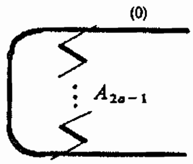
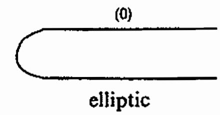
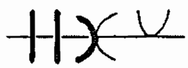
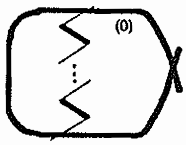
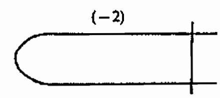
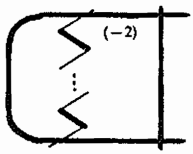
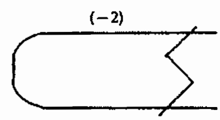
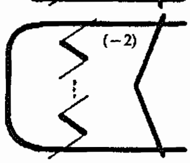
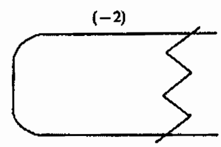
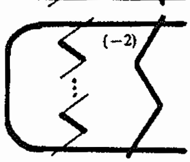
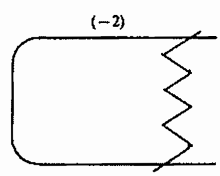
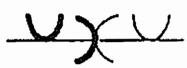
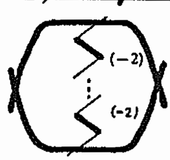
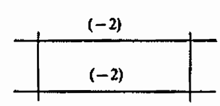
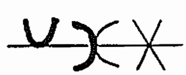
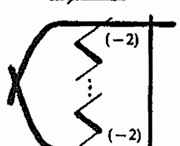
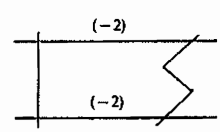
$B_Y(L)$: bold curves,
( ): self-intersection

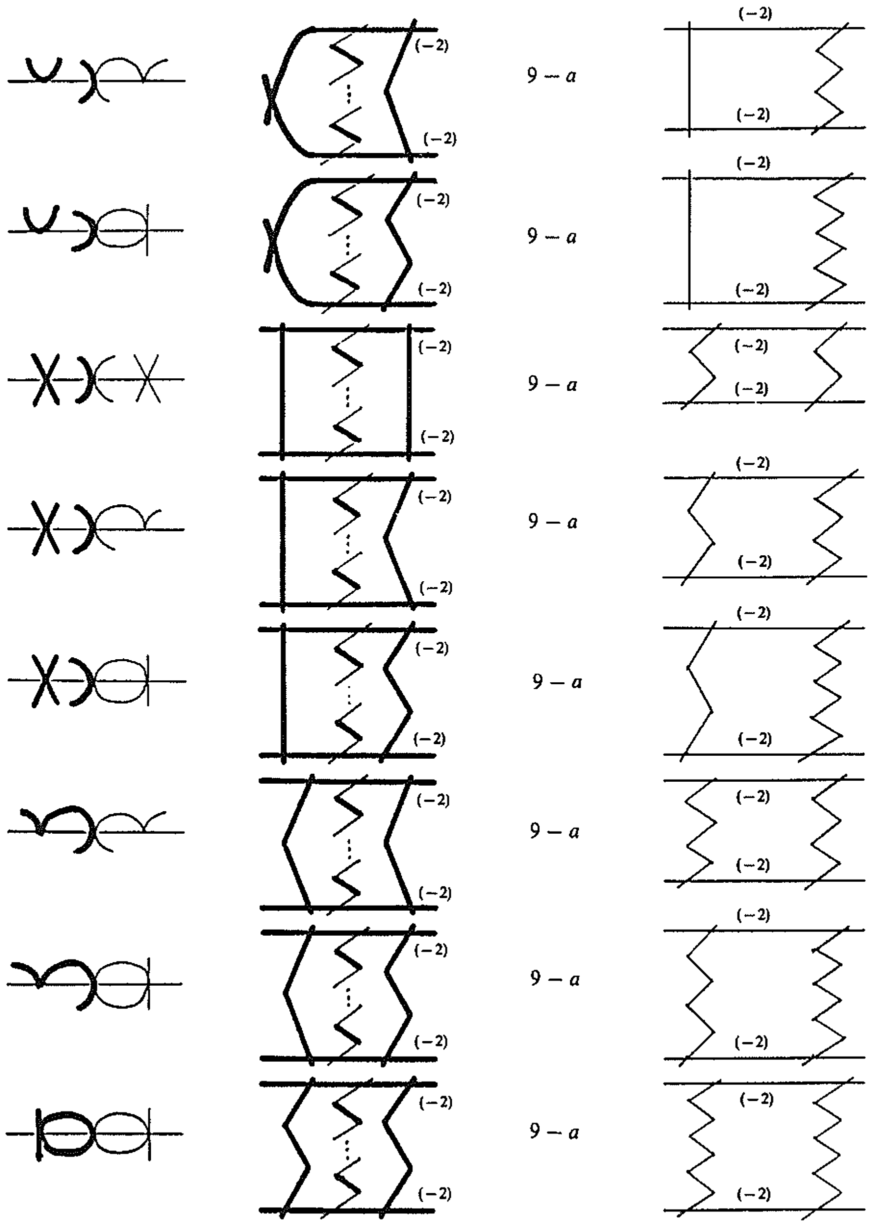
$\# B_Y(I)$

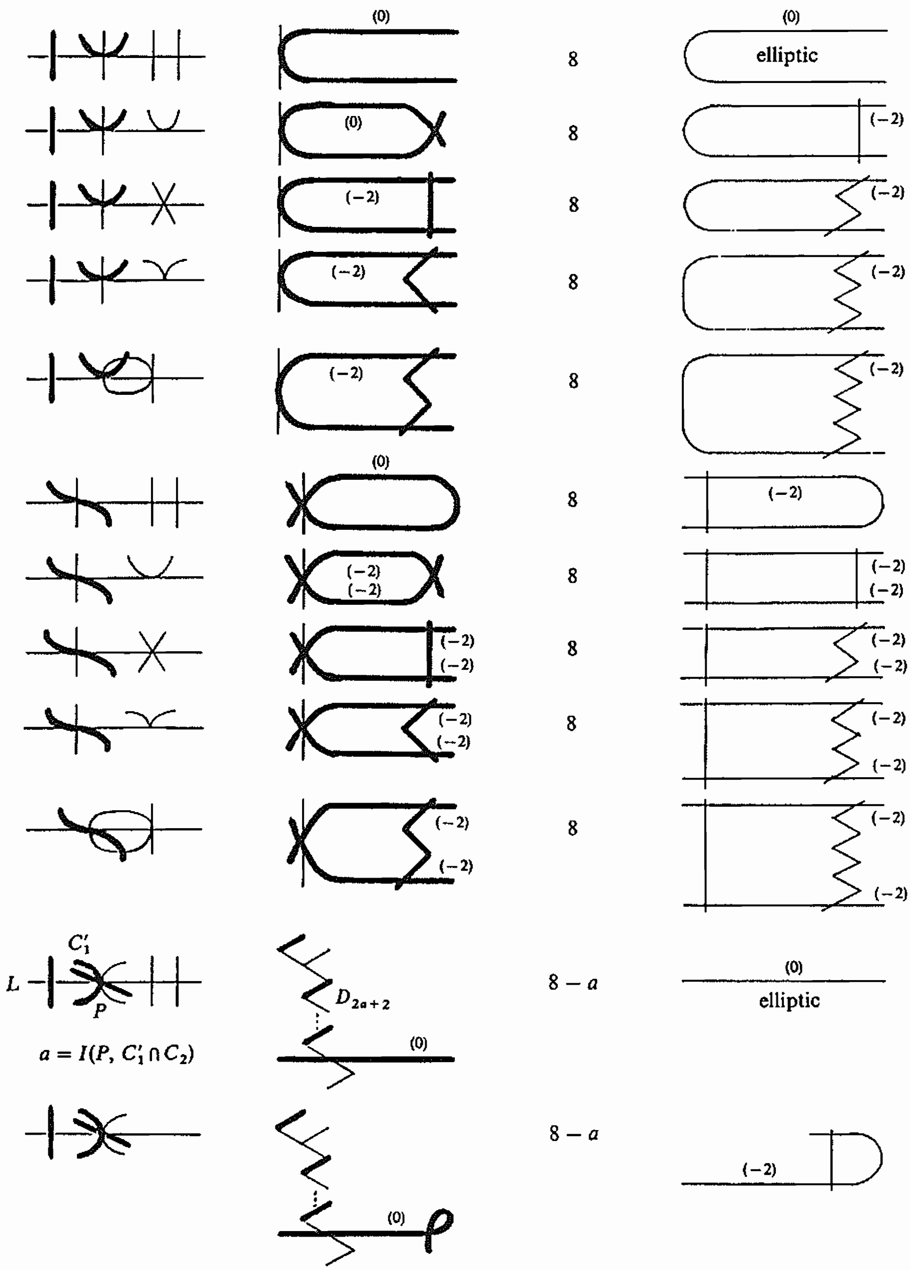
$K_{\hat{X}}$
( ): self-intersection



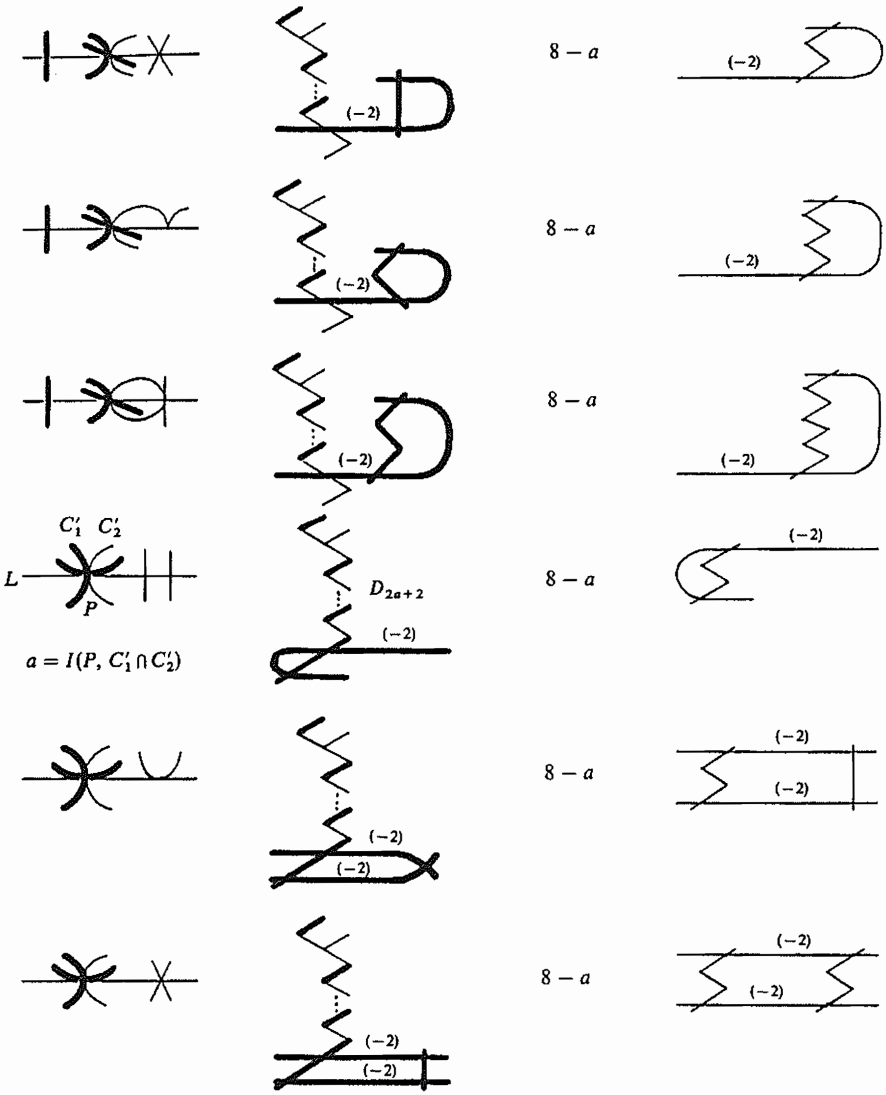
(5.2.3) Case $m(\sum C_j, L) = 1$ (55 types):

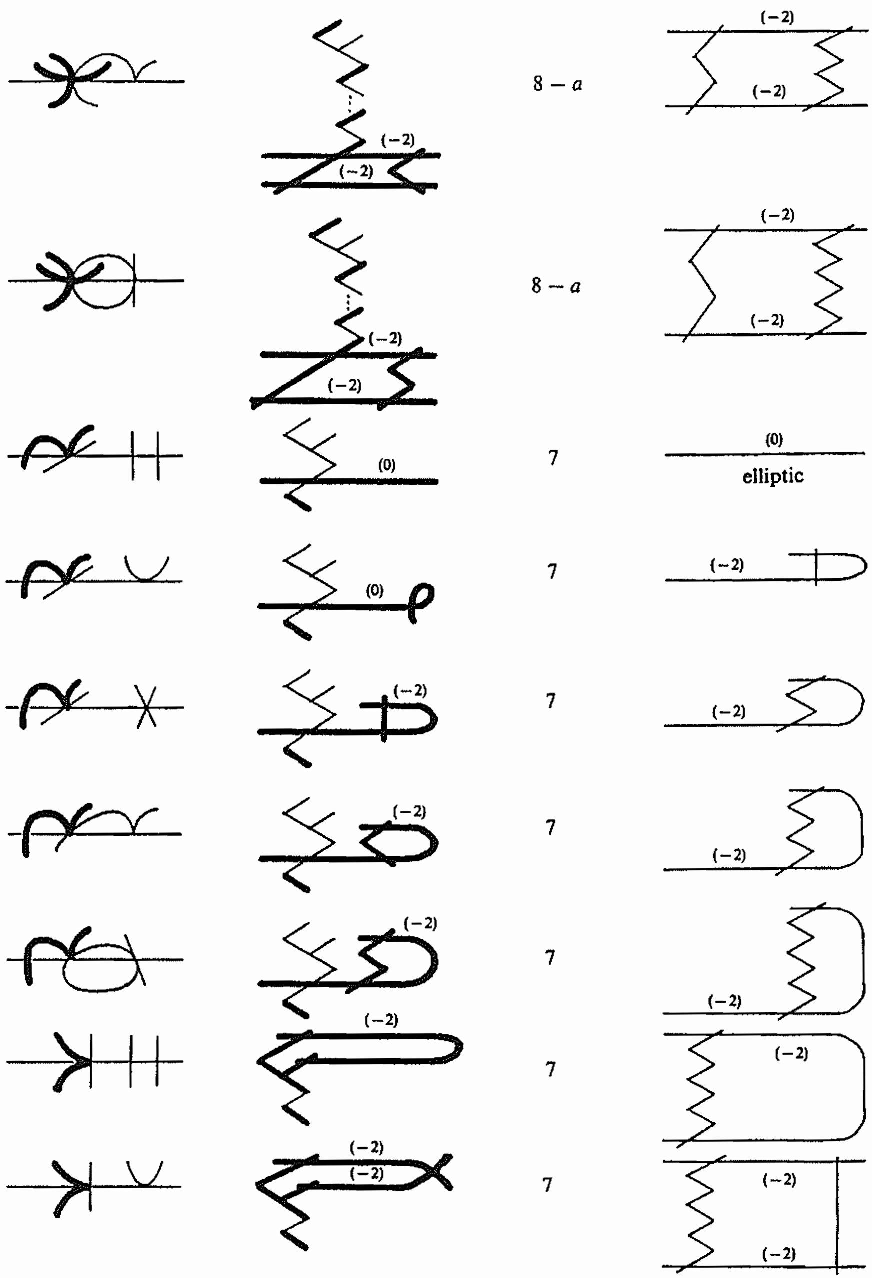
on P^2, C_1: bold curves	$B_Y(L)$: bold curves, ( ): self-intersection	$\# B_Y(I)$	$K_{\hat{X}}$ ( ): self-intersection
 $a = I(P, C_1 \cap C_2)$		$9 - a$	
		$9 - a$	
		$9 - a$	
		$9 - a$	
		$9 - a$	
		$9 - a$	
		$9 - a$	

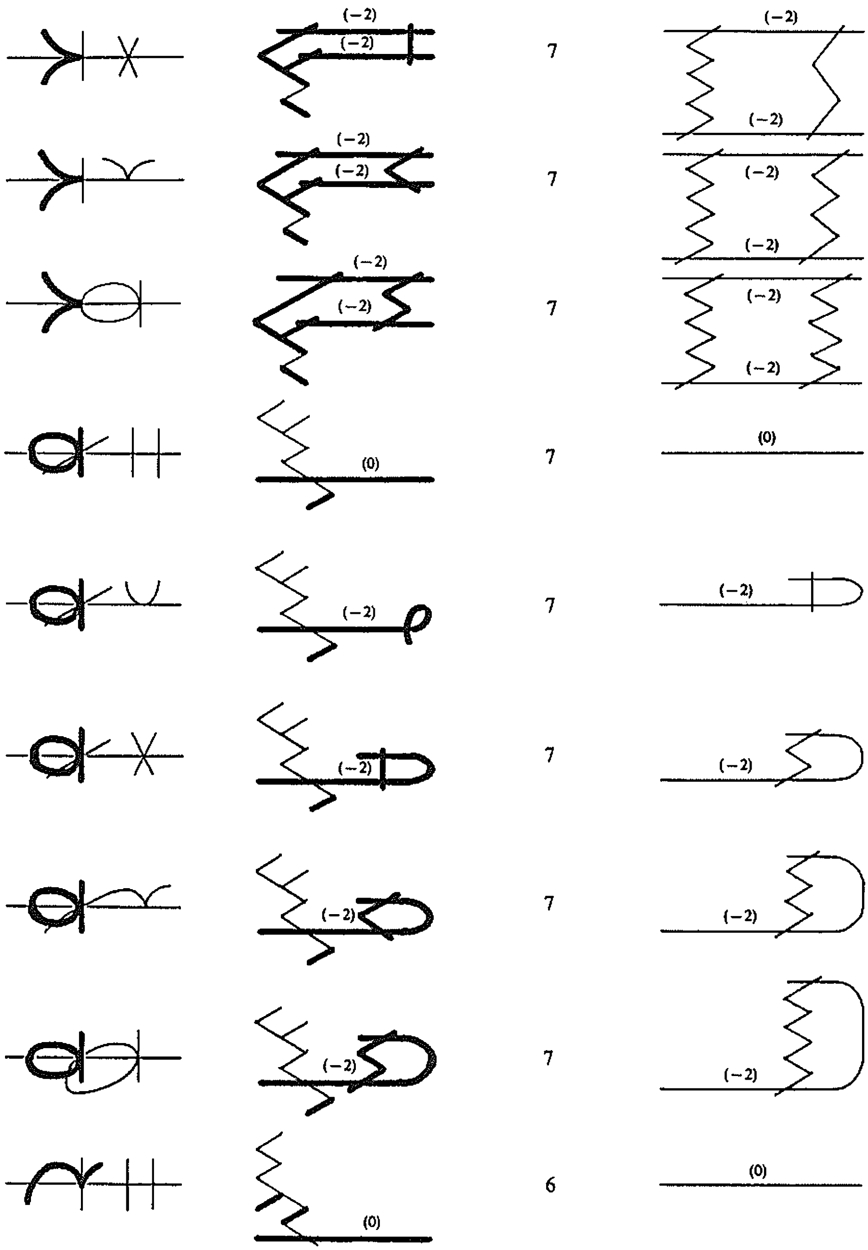

(−2)
(−2)
9 − a
(−2)
(−2)
(−2)
(−2)
9 − a
(−2)
(−2)
(−2)
(−2)
9 − a
(−2)
(−2)
(−2)
(−2)
9 − a
(−2)
(−2)
(−2)
(−2)
9 − a
(−2)
(−2)
(−2)
(−2)
9 − a
(−2)
(−2)
(−2)
(−2)
9 − a
(−2)
(−2)
(−2)
(−2)
9 − a
(−2)
(−2)

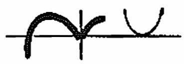
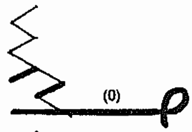
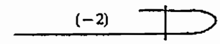
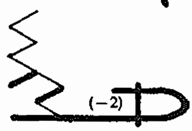
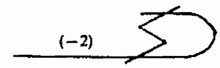
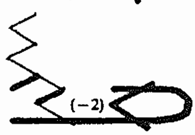
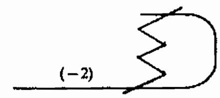
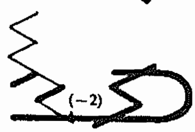
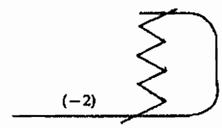


(0)
8
(0)
elliptic
(0)
8
(−2)
(−2)
8
(−2)
(−2)
8
(−2)
(−2)
8
(−2)
(0)
8
(−2)
(−2)
(−2)
8
(−2)
(−2)
(−2)
(−2)
8
(−2)
(−2)
(−2)
(−2)
8
(−2)
(−2)
(−2)
(−2)
8
(−2)
(−2)
C_1'
L
P
D_{2a+2}
(0)
$8-a$
(0)
elliptic
$a = I(P, C_1' \cap C_2)$
$8-a$
(0)
(−2)


(−2)
8 − a
(−2)
(−2)
8 − a
(−2)
(−2)
8 − a
(−2)
C′1 C′2
L
P
a = I(P, C′1 ∩ C′2)
D2a+2
(−2)
8 − a
(−2)
(−2)
(−2)
8 − a
(−2)
(−2)
(−2)
(−2)
8 − a
(−2)
(−2)

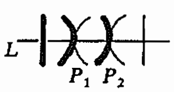
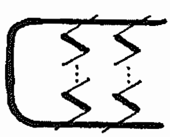
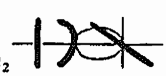
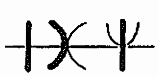
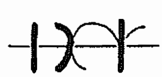
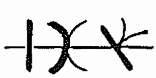
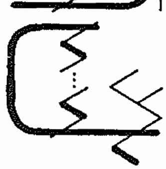
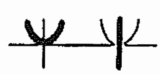
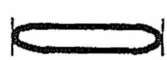
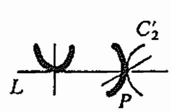
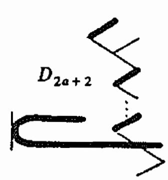

(−2)
(−2)
8 − a
(−2)
(−2)
(−2)
(−2)
8 − a
(−2)
(−2)
(0)
7
(0)
elliptic
(0)
7
(−2)
(−2)
7
(−2)
(−2)
7
(−2)
(−2)
7
(−2)
(−2)
7
(−2)
(−2)
(−2)
7
(−2)
(−2)

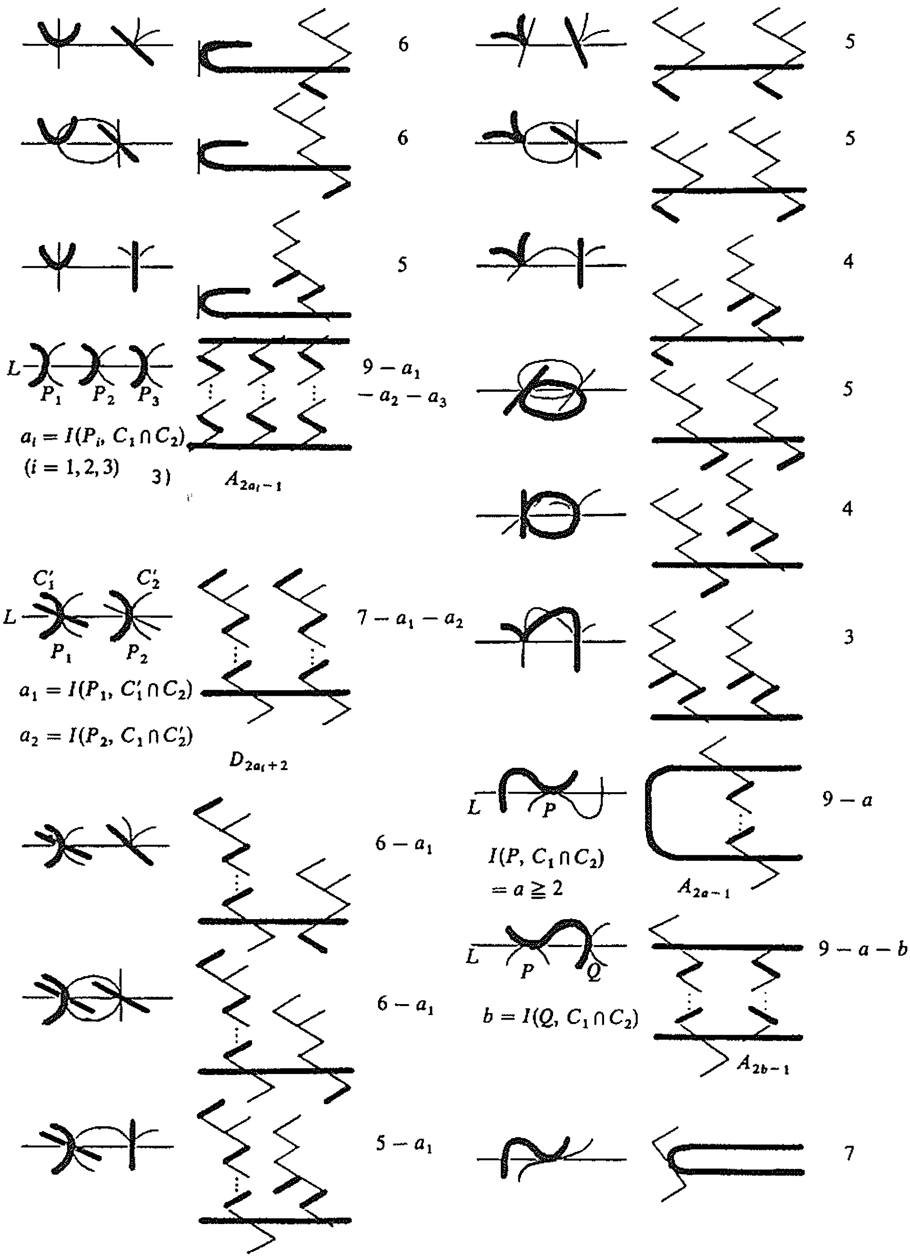

(−2)
(−2)
7
(−2)
(−2)
(−2)
(−2)
7
(−2)
(−2)
(−2)
(−2)
7
(−2)
(−2)
(0)
7
(0)
(−2)
7
(−2)
(−2)
7
(−2)
(−2)
7
(−2)
(−2)
7
(−2)
(0)
6
(0)

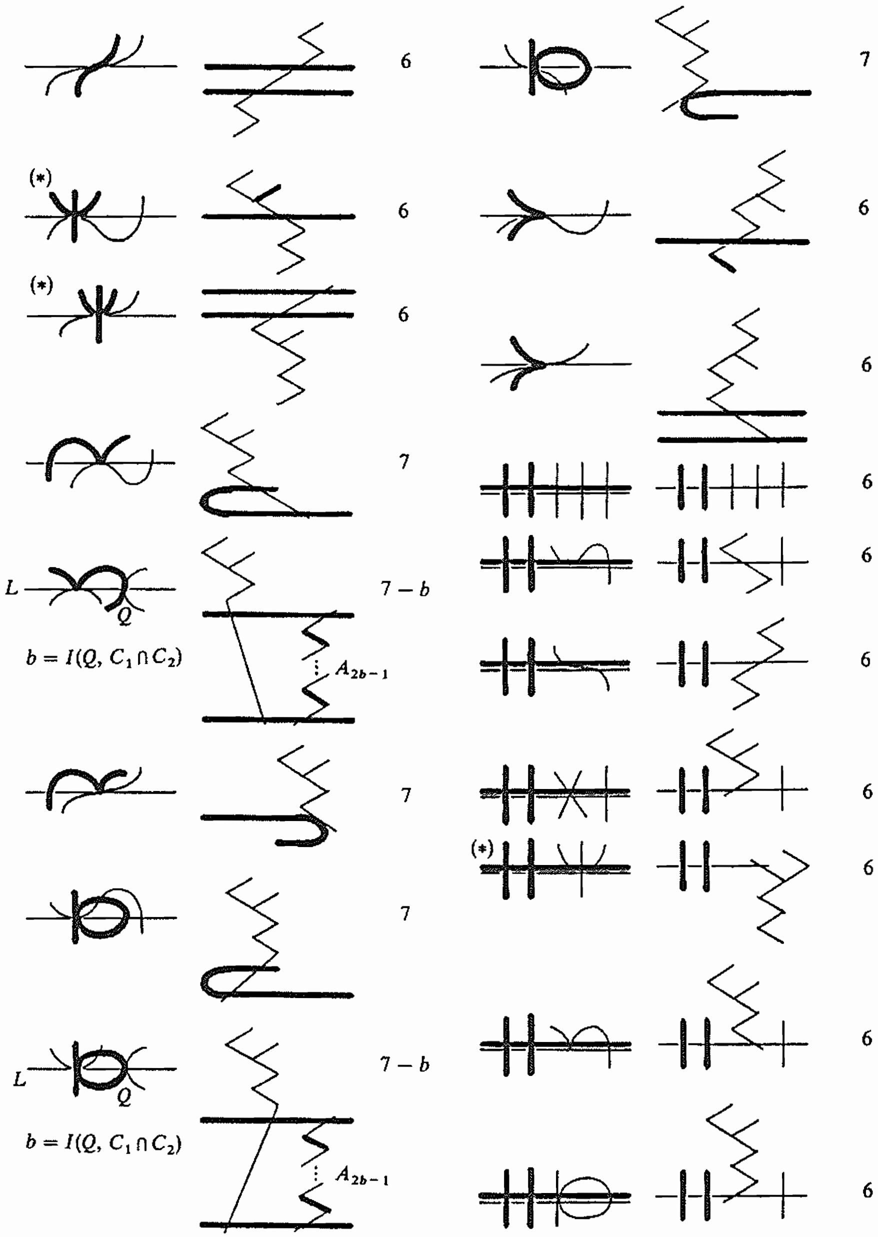
		6	
		6	
		6	
		6	

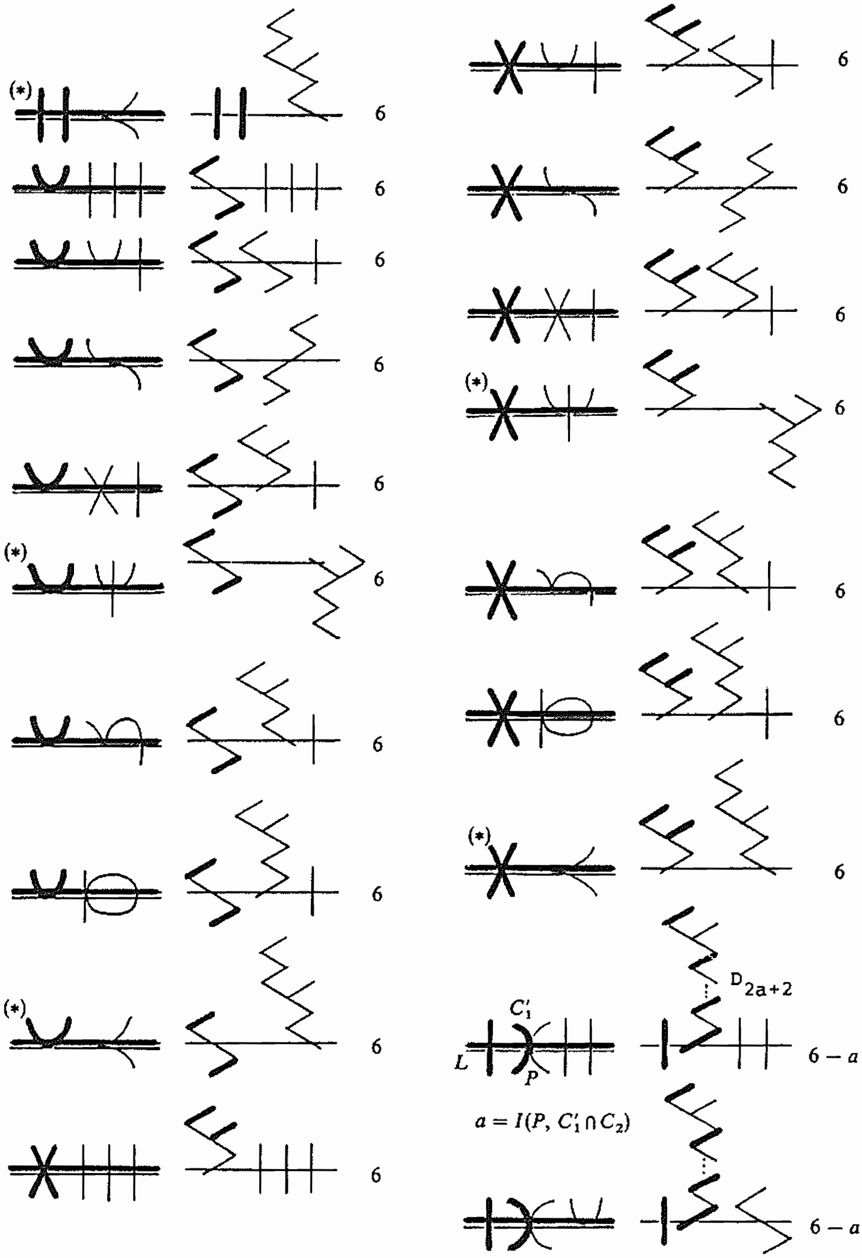
(5.2.4) Case $m(\sum C_j, L) \geqq 2$ (69 types):

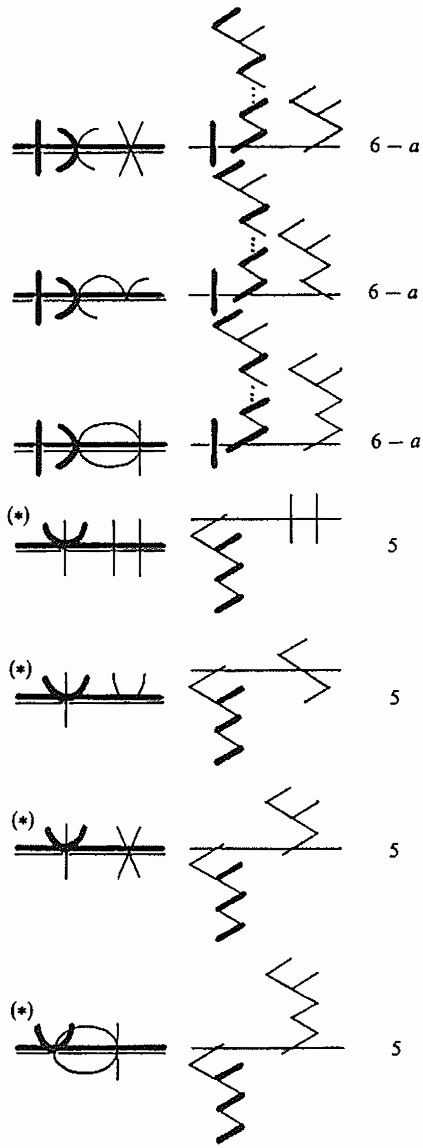
In this case, all curves in $B_Y(L)$ are (-2)-curves hence $K_{\hat{X}}$ is 0.

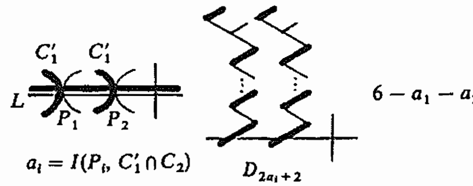
on P^2, C_1: bold curves	$B_Y(L)$: bold curves, ( ): self-intersection	$\# B_Y(l)$	on P^2, C_1: bold curves	$B_Y(L)$: bold curves, ( ): self-intersection	$\# B_Y(l)$
 $a_i = I(P_i, C_1 \cap C_2)$ $(i = 1, 2)$	 A_{2a_i-l}	$9 - a_1 - a_2$			$7 - a_1$
		$8 - a_1$			$6 - a_1$
		$7 - a_1$			7
			 $a = I(P, C_1 \cap C'_2)$		$7 - a$


6
6
5
L
P_1 P_2 P_3
$9 - a_1 - a_2 - a_3$
$a_i = I(P_i, C_1 \cap C_2)$
$(i = 1, 2, 3)$
3)
A_{2a_i-1}
C_1' C_2'
L
P_1 P_2
$7 - a_1 - a_2$
$a_1 = I(P_1, C_1' \cap C_2)$
$a_2 = I(P_2, C_1 \cap C_2')$
D_{2a_i+2}
$6 - a_1$
$6 - a_1$
$5 - a_1$
5
5
4
5
4
3
L
P
$9 - a$
$I(P, C_1 \cap C_2)$
$= a \geqq 2$
A_{2a-1}
L
P Q
$9 - a - b$
$b = I(Q, C_1 \cap C_2)$
A_{2b-1}
7


6
7
(*)
6
6
(*)
6
6
7
6
L
Q
7 − b
6
$b = I(Q, C_1 \cap C_2)$
A_{2b-1}
6
7
6
(*)
6
7
L
Q
7 − b
6
$b = I(Q, C_1 \cap C_2)$
A_{2b-1}
6


(*)
6
D_{2a+2}
C_1'
L
P
$6 - a$
$a = I(P, C_1' \cap C_2)$





$a_i = I(P_i, C'_1 \cap C_2)$
$(i = 1, 2)$

(Curves with (*) are unstable as plane curves of degree 7, cf. [Se].)

(5.3) As a consequence of the above classification of the branch locus B_Y on Y and the canonical curve $K_{\hat{X}}$ of $\hat{X}$, we get another proof of Main Theorem:

PROOF OF THEOREM (2.6.3). We keep the above notation and the notation in (2.3). The case $t \in S_0 \cap T_0$ is already settled in Proposition (3.4). In all cases, $p_g(\hat{X}) = 1$ by Corollary (3.4.1). By Lemma (1.1.2), $c_1^2(\hat{X})$ and $q(\hat{X})$ can be computed from the result of the above classification of B_Y. $K_{\hat{X}}$ is always connected and not multiple by Observation (3.3.1) or by the above result of classification. By construction, we see $\dim|2K_{\hat{X}}| = 2 - \max\{m, n\}$ for $t \in S_m \cap T_n$. This together with the value of $c_1^2(\hat{X})$ determines $\kappa(\hat{X})$. Thus we get:

Case	$\kappa(\hat{X})$	$c_1^2(\hat{X})$	$K_{\hat{X}}$	$p_g(\hat{X})$	$q(\hat{X})$	Type of $\hat{X}$
$t \in S_0 \cap T_0$	2	1	nef and big	1	0	Kunev
$t \in S_1$	1	0	connected and not multiple	1	0	num. K3 with one double fiber
$t \in S_2$	0	0	0	1	0	K3
$t \in T_1 \cap S_0$	1	0	connected and not multiple	1	1	ellip. with $p_g = q = 1$
$t \in T_2$	0	0	0	1	2	abelian

Here we use (1.3.3) and the canonical bundle formula (1.3.1) for determination of the type of $\hat{X}$ in case $\kappa(\hat{X}) = 1$. In fact, for an elliptic fibration $f: \hat{X} \to \Delta$ with multiple fibers $m_i F_i$, (1.3.1) says

$$K_{\hat{X}} = f^*Z + \sum (m_i - 1)F_i$$

for a divisor Z on the base curve Δ with

$$\deg Z = \chi(\mathcal{O}_{\hat{X}}) - 2\chi(\mathcal{O}_\Delta) = \begin{cases} 2g(\Delta) & \text{in case } t \in S_1\,, \\ 2g(\Delta) - 1 & \text{in case } t \in T_1 \cap S_0\,. \end{cases}$$

Since $K_{\hat{X}}$ is connected and not multiple, we have the only possibility for the type of elliptic fibration:

In case $t \in S_1$, there exists unique double fiber and the base curve is rational.

In case $t \in T_1 \cap S_0$, $K_{\hat{X}}$ is a fiber and the base curve is elliptic. Q.E.D.

References

[Us.6] Usui, S., Type I degeneration of Kunev surfaces, to appear in Astérisque, Colloque Théorie de Hodge, Luminy 1987.

Kochi University
Department of Mathematics
Faculty of Science
Kochi, 780 Japan